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2D and 3D exact shock wave solutions with specular reflection to the discrete Boltzmann models

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Abstract. We construct exact solutions which are 2D (two spatial coordinates) for the four velocity planar model, 3D for the cubic Broadwell models and p-dimensional for the p-dimensional hypercubic discrete Boltzmann model. These solutions are shock waves which satisfy a specular reflection boundary condition at a wall.

1. Introduction

For the discrete Boltzmann models [1]‡ the velocity can only take discrete values $\boldsymbol{v}_i, |\boldsymbol{v}_i| = 1, i = 1, \ldots, 2p$. To each velocity $\boldsymbol{v}_i$ is associated a density N_i. Recently $(2+1)$-dimensional solutions [2] (two spatial coordinates plus time) have been constructed for three models: the planar $4\boldsymbol{v}_i$ model, the cubic Broadwell model and a generalised p-dimensional hypercubic model.

Here, in addition we introduce boundary conditions: namely specular reflection at a wall. For the $4\boldsymbol{v}_i$ model we obtain 2D solutions and for the Broadwell 3D. We emphasise that for the first time exact solutions are constructed to the discrete Boltzmann models with three independent spatial coordinates. However, the time dependence is simple and corresponds to a translation of the initial-values patterns. As usual the solutions are sums of similarity shock waves [3].

In § 2 we study the $4\boldsymbol{v}_i$ model and in § 3 the Broadwell model. In § 4 we introduce the hypercubic p-dimensional model, which for $p = 2, 3$ is reduced to the previous models, and we construct p-dimensional exact solutions.

2. 2D solutions for the $4\boldsymbol{v}_i$ planar model

For this model, with four velocities $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_3 + \boldsymbol{v}_4 = 0$, $\boldsymbol{v}_1$, $\boldsymbol{v}_3$ lying along the x_1 and x_2 positive axes respectively, the equations for the four densities N_i are

$$N_{1t} + N_{1x_1} = N_{2t} - N_{2x_1} = -N_{3t} - N_{3x_2} = -N_{4t} + N_{4x_2} = N_3N_4 - N_1N_2. \tag{2.1}$$

We start with an ansatz:

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} = \begin{pmatrix} n_{01} \\ n_{02} \\ n_{01} \\ n_{02} \end{pmatrix} + \begin{pmatrix} n_1 & n_3 \\ n_1 & n_4 \\ n_3 & n_1 \\ n_4 & n_1 \end{pmatrix} \begin{pmatrix} 1/D_2 \\ 1/D_1 \end{pmatrix} \qquad D_i = 1 + d\exp(\rho t + \gamma x_i) \quad d > 0. \tag{2.2}$$

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‡ Note that only models with more than $2v_i$ can lead to multidimensional solutions.

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The conditions $N_1 = N_3$, $N_2 = N_4$ for a specular reflection at a wall $x_1 = x_2$ are satisfied. Substituting (2.2) into (2.1) we write down the relations for the coefficients of D_i^{-1}, D_i^{-2}:

$$n_1\rho = -n_3(\rho+\gamma) = n_4(-\rho+\gamma) = n_3n_4 - n_1^2 = n_{01}(n_1 - n_4) + n_{02}(n_1 - n_3). \tag{2.3}$$

In the collision term of (2.1) the constant and the terms proportional to $(D_1D_2)^{-1}$ are identically zero. We have seven parameters n_i, n_{0i}, ρ, γ and four relations, leaving three arbitrary parameters. The frequency ρ and wavenumber γ can be obtained from the n_i which satisfy a closed relation:

$$\gamma = \rho(n_4 - n_3)/(n_4 + n_3) \qquad \rho = -n_1 + n_3n_4/n_1 \qquad -2n_3n_4 = n_1(n_3 + n_4). \tag{2.4}$$

For the construction of the solutions we define intermediate parameters $\bar{n}_i = n_i/n_1$, $i = 3, 4$. We choose for the arbitrary parameters

$$S = \bar{n}_3 + \bar{n}_4 > 0 \qquad n_{01} > 0 \qquad n_{02} > 0 \tag{2.5}$$

from which we will construct all others. From (2.4) $2\bar{n}_3\bar{n}_4 = -S$, and it follows that the $\bar{n}_i$ are known:

$$2\bar{n}_3 = S + \sqrt{S(S+2)} > 0 \qquad 2\bar{n}_4 = S - \sqrt{S(S+2)} < 0. \tag{2.6}$$

n_1 is found from the last relation of (2.3):

$$n_1 = \frac{2}{(S+2)}[n_{01}(\bar{n}_4 - 1) + n_{02}(\bar{n}_4 - 1)]. \tag{2.7}$$

Finally from $\bar{n}_i$, n_1, which are functions of S, n_{01}, n_{02}, we obtain $n_i = \bar{n}_i n_1$, $i = 3,4$, ρ and γ.

Physically relevant densities must be positive. It is sufficient that positivity holds for the asymptotic shock limits at $t = 0$ in the x_1, x_2 plane. These limits are plateaus in the x_1, x_2 coordinate plane and four for each density. However, the plateaus for N_1, N_3 and N_2, N_4 being the same, only eight are independent:

$$\begin{array}{llll} n_{01} & \Sigma_{11} = n_{01} + n_1 & \Sigma_{13} = n_{01} + n_3 & \Sigma_{113} = n_{01} + n_1 + n_3 \\ n_{02} & \Sigma_{21} = n_{02} + n_1 & \Sigma_{24} = n_{02} + n_4 & \Sigma_{214} = n_{02} + n_1 + n_4. \end{array} \tag{2.8}$$

Let us assume that all these limits are positive (or at least non-negative); then, for instance,

$$N_1D_1D_2 = y_1y_2n_{01} + y_2\Sigma_{13} + y_1\Sigma_{11} + \Sigma_{113} > 0 \qquad y_i = d\,\exp(\rho t + \gamma x_i) > 0. \tag{2.8'}$$

It remains to find the domain of the arbitrary parameter space leading to $\Sigma_i > 0$. We rewrite these limits with the arbitrary parameters:

$$\begin{array}{lll} \Sigma_{11} = \Omega_1(n_{02} - n_{01}\bar{n}_4) & \Sigma_{13} = \Omega_1(n_{02}\bar{n}_3 - n_{01}) & \Omega_1 = 2(\bar{n}_3 - 1)/(S+2) \\ \Sigma_{21} = \Omega_2(n_{02}\bar{n}_3 - n_{01}) & \Sigma_{24} = \Omega_2(n_{02} - \bar{n}_4 n_{01}) & \Omega_2 = 2(1 - \bar{n}_4)/(S+2) \\ \Sigma_{113} = \Omega_3[n_{01}(\bar{n}_4 - \bar{n}_3) + n_{02}(\bar{n}_3^2 - 1)] & & \\ \Sigma_{214} = \Omega_3[n_{01}(\bar{n}_4^2 - 1) + n_{02}(\bar{n}_3 - \bar{n}_4)] & \Omega_3 = 2/(S+2) > 0 \quad \bar{n}_3 > 0 \quad \bar{n}_4 < 0 & \end{array} \tag{2.9}$$

Lemma 1. All $\Sigma_i > 0$ if $S > 2/3$ and if $n_{02} > n_{01}\sup(B_1 = 1/\bar{n}_3, B_2 = (\bar{n}_3 - \bar{n}_4)/(\bar{n}_3^2 - 1), B_3 = (1 - \bar{n}_4^2)/(\bar{n}_3 - \bar{n}_4))$.

First we notice from (2.6) that $|\bar{n}_4| < 1$ and $\bar{n}_3 > 1$ if $S > 2/3$. It follows that $\Omega_1 > 0, \Omega_2 > 0, B_2 > 0, B_3 > 0, \Sigma_{11} > 0$ and $\Sigma_{13} > 0$ if $n_{02} > n_{01}/\bar{n}_3$. Further we find $\Sigma_{113} > 0$ if $n_{02} > B_2 n_{01}$ and $\Sigma_{214} > 0$ if $n_{02} > B_3 n_{01}$.

Lemma 2. If $S > 2/3$ then $B_2 > B_1$, $B_2 > B_3$.

This follows from $B_2 - B_1 = (S+2)/2\bar{n}_3(\bar{n}_3^2 - 1) > 0$ and $B_2 - B_3 = (2+S)^2/4(\bar{n}_3 - \bar{n}_4)(\bar{n}_3^2 - 1)$ and we deduce the following theorem.

Theorem. All N_i are positive if $S > 2/3$, $n_{0i} > 0$ and if $n_{02}/n_{01} > (\bar{n}_3 - \bar{n}_4)/(\bar{n}_3^2 - 1)$, the lower bound being S dependent.

Next we introduce the total mass $M = \Sigma N_i$ which from (2.1.7) can be written

$$M = m_0 + m(1/D_1 + 1/D_2) \qquad m_0 = 2(n_{01} + n_{02})$$
$$m = n_1(2+S) = 2[n_{01}(\bar{n}_4 - 1) + n_{02}(\bar{n}_3 - 1)] = -2\rho. \tag{2.10}$$

We find $m \gtrless 0$ if $n_{02} \gtrless n_{01} B_4$, $B_4 = (1-\bar{n}_4)/(\bar{n}_3 - 1) > B_2$. It follows that $m, \rho, \gamma = -\rho\sqrt{(S+2)/S}$ can be positive or negative but m, γ have the same sign and ρ the opposite one. For the shock speed $c = \rho/\gamma$ we find $|c| < 1$. The time dependence is very simple. At fixed time t

$$M(x_1, x_2; t) = M(x_1 + ct, x_2 + ct; 0) \qquad c = \rho/\gamma \tag{2.11}$$

corresponds in the coordinate x_1, x_2 plane to a translation ct of the initial data. The construction at fixed t of the equidensity lines $M(x_1, x_2; t) = \text{constant} = C$ in the x_1, x_2 plane is easy and we find for x_2 as a simple function of x_1:

$$x_2 = \frac{1}{\gamma} \log\left[\frac{1}{\bar{d}}\left(-1 + \frac{m}{C - m_0 - m[1 + \bar{d}\exp(\gamma x_1)]^{-1}}\right)\right] \qquad \bar{d} = d\exp(\rho t)$$

where the square bracket must be positive. In the x_1, x_2 plane the four asymptotic plateaus (three for both domains at the right and at the left of the wall $x_1 = x_2$) are m_0, $m_0 + m$ (twice) and $m_0 + 2m$. If $m < 0$ (or $\rho > 0$) the upstream plateau is the Maxwellian m_0, while the downstream one is $m_0 + 2m$, and the converse if $m > 0$ ($\rho < 0$).

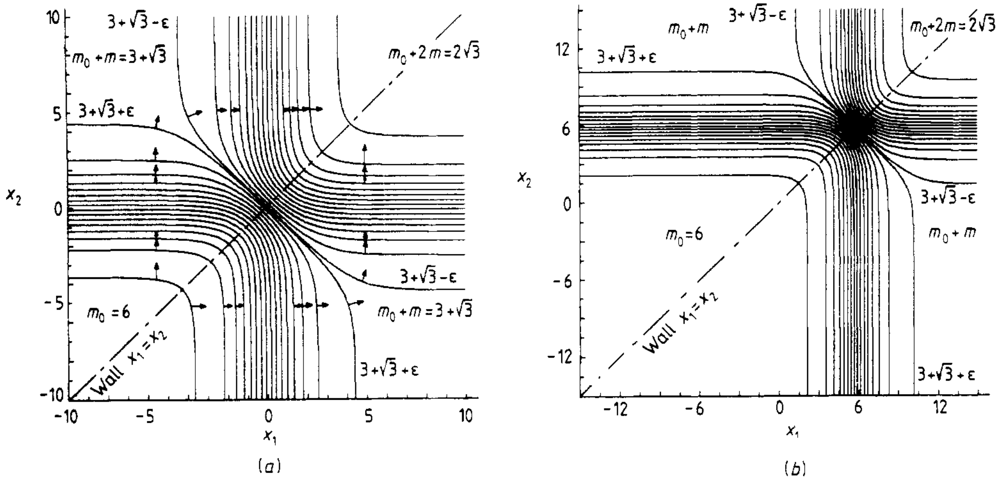
As illustration, we choose the simple example in figure 1:

$$n_{01} = 1 \qquad n_{02} = 2 \qquad S = 1 \qquad d = 1 \tag{2.12}$$

from which we deduce:

$$2\bar{n}_i = 1 \pm \sqrt{3} \qquad n_1 = -1 + 1/\sqrt{3} \qquad \sqrt{3}n_3 = -1 \qquad n_4 = -1 + 2/\sqrt{3}$$
$$2\rho = 3 - \sqrt{3} > 0 \qquad \gamma = -\sqrt{3}\rho < 0 \qquad \sqrt{3}c = -1 \qquad |c| < 1$$
$$m_0 = 6 \qquad m = \sqrt{3} - 3 < 0 \qquad m_0 + 2m = 2\sqrt{3} \qquad m_0 + m = 3 + \sqrt{3}.$$

All Σ_i are positive except $\Sigma_{113} = 0$, but from (2.8′) we see that $N_1 \geqslant 0$ does not violate positivity. In figure 1(a) we present the M equidensity lines at $t = 0$. The arrays represent decreasing equidensity lines. The upstream plateau $m_0 = 6$, due to $\rho > 0$, is also the Maxwellian one. The downstream plateau is $m_0 + 2m = 2\sqrt{3}$ and we observe the two intermediate plateaus $m_0 + m = 3 + \sqrt{3}$. The equidensity lines are perpendicuar to the wall $x_1 = x_2$ and they are symmetric with respect to the wall. Along any profile parallel to the wall we observe the shock domain between upstream and downstream plateau. We see one shock in figure 1(c) along the wall $x_1 = x_2$ while at some distance $x_2 = x_1 - 15$ from the wall such profiles exhibit a double shock with $m_0 + m$ as an intermediate plateau. Along profiles parallel either to the x_1 axis or x_2 axis we observe a simple shock either from m_0 toward $m_0 + m$ or from $m_0 + m$ toward $m_0 + 2m$. In



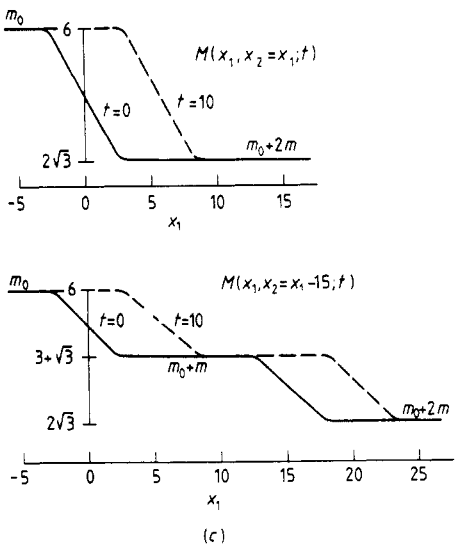


Figure 1. Illustration of the $4v_i$ model. (a) Equidensity lines at $t=0$, $M(x_1, x_2; t=0)=$ constant. (b) Equidensity lines at $t=10$, $M(x_1, x_2; t=10)=$ constant. In both (a) and (b) $\varepsilon = 10^{-2}$. (c) Shock profiles with one and two shocks connecting two and three plateaus, respectively.

figure 1(b, c, d) for the equidensity M lines at $t=10$, we observe a translation $-ct = 10/\sqrt{3}$ of the $t=0$ profiles. This can be interpreted by the fact that the Maxwellian plateau $m_0=6$ spreads out when the time increases.

Strictly speaking the equidensity lines $M=m_0+m$, with the (2.12) choice $d=1$, are provided by the lines $x_1+x_2+2ct=0$. However, the δM increment for large $|x_i|$ is very small around that line. In figure 1(a, b), in order to figure out the m_0+m plateaus, we present equidensity lines either between $m_0+m-\varepsilon$ and m_0+2m or between $m_0+m+\varepsilon$ and m_0 with the same increment δM for both sets. We choose $\varepsilon=10^{-2}$ and between the two sets the shift is $2\varepsilon \neq \delta M$.

3. 3D solutions for the $6v_i$ cubic Broadwell model

For this model with six velocities $v_1+v_2=v_3+v_4=v_5+v_6=0$, v_1, v_3, v_5 lying along the

x_1, x_2, x_3 positive axes respectively, the equations for the six densities N_i are

$$
\begin{aligned}
&N_{1t} + N_{1x_1} = N_{2t} - N_{2x_1} = N_3N_4 + N_5N_6 - 2N_1N_2 \\
&N_{3t} + N_{3x_2} = N_{4t} - N_{4x_2} = N_5N_6 + N_1N_2 - 2N_3N_4 \\
&N_{5t} + N_{5x_3} = N_{6t} - N_{6x_3} = -N_{1t} - N_{1x_1} - N_{3t} - N_{3x_2}.
\end{aligned}
\tag{3.1}
$$

We start with an ansatz which is an obvious generalisation of (2.2):

$$
\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{pmatrix} = \begin{pmatrix} n_{01} \\ n_{02} \\ n_{01} \\ n_{02} \\ n_{01} \\ n_{02} \end{pmatrix} + \begin{pmatrix} n_1 & n_5 & n_1 \\ n_1 & n_6 & n_1 \\ n_1 & n_1 & n_5 \\ n_1 & n_1 & n_6 \\ n_5 & n_1 & n_1 \\ n_6 & n_1 & n_1 \end{pmatrix} \begin{pmatrix} 1/D_3 \\ 1/D_1 \\ 1/D_2 \end{pmatrix} \qquad D_i = 1 + d(\rho t + \gamma x_i) \quad d > 0. \tag{3.2}
$$

The conditions $N_1 = N_3$, $N_2 = N_4$ for a specular reflection at a plane wall $x_1 = x_2$ are satisfied (notice that $\boldsymbol{v}_5$ and $\boldsymbol{v}_6$ being parallel to the wall, no conditions on N_5, N_6 occur). Substituting (3.2) into (3.1) we first write down the relations for the coefficients of D_i^{-1}, D_i^{-2}:

$$
\begin{aligned}
&n_1\rho = n_5n_6 - n_1^2 = n_1(n_{01} + n_{02}) - n_5n_{02} - n_6n_{01} \\
&n_5(\rho + \gamma) = n_6(\rho - \gamma) = -2n_1\rho.
\end{aligned}
\tag{3.3a}
$$

Again, the constants in the collision terms of (3.1) are identically zero, but the coefficients of $(D_pD_q)^{-1}$, $p \neq q$, give a new relation

$$
n_5 + n_6 = 2n_1. \tag{3.3b}
$$

We still have seven parameters n_i, n_{0i}, ρ, γ but here five relations, leaving only two arbitrary parameters. Frequency ρ and wavenumber γ are still deduced from the n_i which satisfy a closed second relation:

$$
\gamma = \rho(n_6 - n_5)/2n_1 \qquad \rho = -3n_1 \qquad n_5n_6 = -2n_1^2. \tag{3.4}
$$

For the construction of the solutions we again define intermediate parameters $\bar{n}_i = n_i/n_1$, $i = 5, 6$, and find $\bar{n}_i = 1 \pm \sqrt{3}$. We choose $\bar{n}_5 = 1 + \sqrt{3}$, $\bar{n}_6 = 1 - \sqrt{3}$ and for the arbitrary parameters

$$
n_{02} > n_{01} > 0. \tag{3.5}
$$

We reconstruct easily all parameters from n_{0i}, $i = 1, 2$:

$$
\begin{aligned}
&\sqrt{3}n_1 = n_{02} - n_{01} > 0 \qquad \sqrt{3}n_5 = (1+\sqrt{3})(n_{02} - n_{01}) > 0 \\
&\sqrt{3}n_6 = (1-\sqrt{3})(n_{02} - n_{01}) < 0 \qquad \rho = \sqrt{3}(n_{01} - n_{02}) < 0 \\
&\gamma = -\sqrt{3}\rho > 0 \qquad c = \rho/\gamma = -1/\sqrt{3} \qquad |c| < 1.
\end{aligned}
\tag{3.6}
$$

For the positivity of the densities N_i we remark that they are sums of positive terms except for the N_i, i even, containing one negative term proportional to n_6. For the even densities we find positive lower bounds:

$$
N_i > n_{02} + n_6/D_j \geq n_{02} + n_6 = (n_{02} + (\sqrt{3} - 1)n_{01})/\sqrt{3} > 0 \qquad (i, j) = (2, 1), (4, 3), (6, 1). \tag{3.7}
$$

In the following we introduce the total mass $M = \Sigma N_i$:

$$M = m_0 + m \sum_1^3 1/D_i \qquad m_0 = 3(n_{01} + n_{02}) > 0$$
$$m = 6n_1 = 2\sqrt{3}(n_{02} - n_{01}) > 0 \tag{3.8}$$
$$D_j = 1 + d\, \exp[3(n_{02} - n_{01})(x_j - t/\sqrt{3})].$$

In the three-dimensional space there exist eight sectors defined by the different signs of each coordinate; for instance, $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, and so on. In these sectors exist asymptotic limits when $|x_j| \to \infty$, $j = 1, 2, 3$, which are volumes and which replace the plateaus of the previous two-dimensional shock waves. There exist eight asymptotic values: m_0, $m_0 + m$ (three times), $m_0 + 2m$ (three times) and $m_0 + 3m$. Due to $m > 0$ the highest asymptotic value is $m_0 + 3m$ in the upstream domain while the lowest one is m_0 in the downstream domain. Since $\rho < 0$, the Maxwellian corresponds to the $m_0 + 3m$ asymptotic limit. Here also the time dependence

$$M(x_1, x_2, x_3; t) = M(x_1 + ct, x_2 + ct, x_3 + ct; 0)$$

corresponds in the x_1, x_2, x_3 space to a translation of the initial $t = 0$ data. Here also the construction of the equidensity lines $M(x_1, x_2, x_3; t) = \text{constant} = C$ (at fixed t) is simple and x_3 is a function of x_1 and x_2:

$$x_3 = \frac{1}{\gamma} \log\left[\frac{1}{\bar{d}}\left(-1 + \frac{m}{C - m_0 - m\Sigma_1^2[1 + \bar{d}\exp(\gamma x_i)]^{-1}}\right)\right] \qquad \bar{d} = d\,\exp(\rho t) \tag{3.9}$$

where the square bracket must be positive.

The present class of solutions (3.2) satisfies a specular reflection boundary condition $N_1 = N_3$, $N_2 = N_4$ at a planar wall $x_1 = x_2$ as well as specular reflection $N_1 = N_5$, $N_2 = N_6$ at a wall $x_3 = x_1$ or specular reflection $N_3 = N_5$, $N_4 = N_6$ at a wall $x_3 = x_2$. Further along the line $x_1 = x_2 = x_3$ we have $N_1 = N_3 = N_5$ and $N_4 = N_6 = N_2$.

As an illustration, in figure 2, we present an example with $n_{01} = 1$, $n_{02} = 2$, $d = 1$ from which we deduce

$$\sqrt{3}n_1 = 1 \qquad n_5 = 1 + 1/\sqrt{3} \qquad n_6 = -1 + 1/\sqrt{3}$$
$$\rho = -\sqrt{3} < 0 \qquad \gamma = 3 > 0 \qquad c\sqrt{3} = -1$$
$$m_0 = 9 \qquad m = 2\sqrt{3} > 0 \qquad 9 \leqslant M \leqslant 9 + 6\sqrt{3}.$$

The M = constant equidensity lines are now surfaces in the three-dimensional x_1, x_2, x_3 space. On a plane we cannot, as was possible in § 2 for the flow of curves M = constant, draw the flow of such surfaces when the constants M are varying. So we choose sections in the space which are either parallel to the wall $x_2 = x_1$ or perpendicular.

In figure 2(a), at $t = 0$, we present the M = constant lines inside the wall $x_2 = x_1$. In such sections of the space, the asymptotic values $m_0 + jm$, $j = 0, 1, 2, 3$, become plateaus. We observe the downstream plateau m_0, the upstream plateau $m_0 + 3m$ (which due to $\rho < 0$ is also the Maxwellian plateau) and the two intermediate $m_0 + m_1$, $m_0 + 2m$ plateaus. In this section the two intermediate plateaus are separated by the shock domains which are two strips parallel either to the x_1 axis or to the x_3 axis. Profiles parallel to x_3 (or x_1) link two plateaus separated by one shock. On the contrary in figure 2(b), at $t = 0$, for a section $x_2 = x_1 + 3$ parallel to the wall, the strip parallel to the x_3 axis is divided in three different parts; two for $x_3 > 0$ and two for $x_3 < 0$. Consequently profiles parallel to x_1 connect three plateaus with two different shocks. In figure 2(c) we present shock profiles with one, two or three shocks connecting two, three or four plateaus.

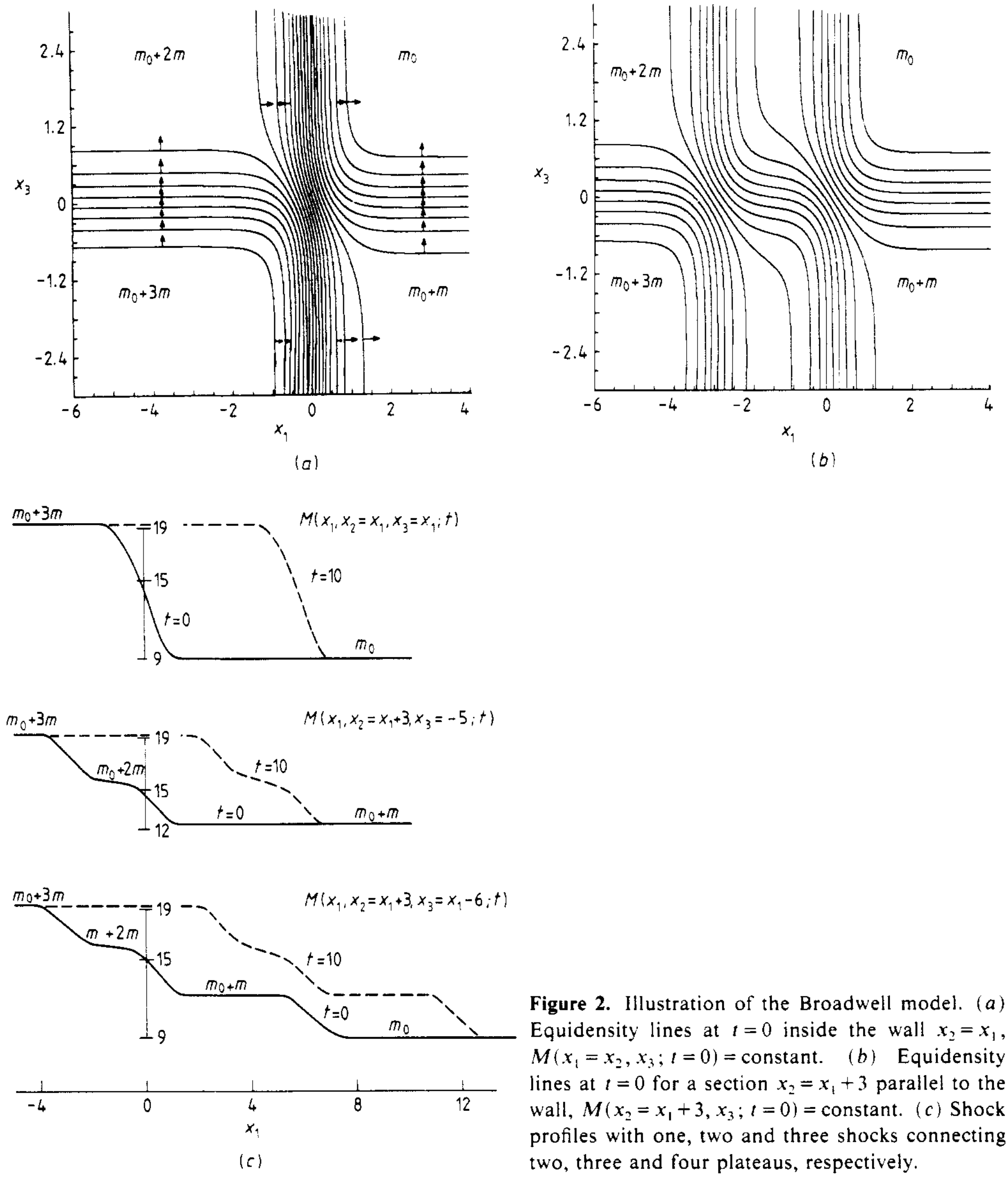


Figure 2. Illustration of the Broadwell model. (*a*) Equidensity lines at $t=0$ inside the wall $x_2=x_1$, $M(x_1=x_2, x_3; t=0)=\text{constant}$. (*b*) Equidensity lines at $t=0$ for a section $x_2=x_1+3$ parallel to the wall, $M(x_2=x_1+3, x_3; t=0)=\text{constant}$. (*c*) Shock profiles with one, two and three shocks connecting two, three and four plateaus, respectively.

For sections perpendicular to the wall, let us choose the simplest one, $x_2+x_1=0$, with the x_3 axis belonging to the wall. At $t=0$ we have $M=m_0+m\,\{1+1/[1+\exp(\gamma x_3)]\}$ and the equidensity lines $M=M(x_3)$ are perpendicular to the wall.

4. *p*-dimensional solutions for the 2*p*-velocity hypercubic model

For this model with $2p$ velocities $\boldsymbol{v}_{2q-1}+\boldsymbol{v}_{2q}=0$, $q=1,2,,\ldots,p$, with $\boldsymbol{v}_{2q-1}$ along the positive x_q axis of a p-dimensional space $x_1, x_2, \ldots x_p$, the equations for the $2p$ densities

N_i, $i = 1, \ldots, 2p$, are

$$N_{2q-1t} + N_{2q-1x_q} = N_{2qt} - N_{2qx_q}$$
$$= -(p-1) N_{2q-1} N_{2q} + \sum_{k \neq q} N_{2k-1} N_{2k} = 0 \qquad q = 1, \ldots, p-1 \tag{4.1}$$
$$N_{2p-1t} + N_{2p-1x_p} = N_{2pt} - N_{2px_p} = -\left(\sum_1^{p-1} (N_{2k-1t} + N_{2k-1x_k}) \right)$$

which reduce to (2.1) for $p = 2$ and (3.1) for $p = 3$.

We start with an ansatz which is an obvious generalisation of both (2.2) and (3.2):

$$N_{2q-1} = n_{01} + n_1 \left(\sum_{k \neq q} 1/D_k \right) + n_{2p-1}/D_q \qquad D_k = 1 + d \exp(\rho t + \gamma x_k)$$
$$N_{2q} = n_{02} + n_1 \left(\sum_{k \neq q} 1/D_k \right) + n_{2p}/D_q \qquad q = 1, 2, \ldots, p \quad d > 0. \tag{4.2}$$

The conditions $N_1 = N_3$, $N_2 = N_4$ for a specular reflection at an hyperplane $x_1 = x_2$ are satisfied. Substituting (4.2) into (4.1) we write down the relations for the coefficients of D_1^{-1}, D_q^{-2}:

$$n_1 \rho = n_{2p-1} n_{2p} - n_1^2 = n_1(n_{01} + n_{02}) - n_{2p-1} n_{02} - n_{2p} n_{01}$$
$$n_{2p-1}(\rho + \gamma) = n_{2p}(\rho - \gamma) = -(p-1) n_{1\rho}. \tag{4.3a}$$

Again, the constants in the collision terms of (4.1) vanish while the coefficients of $(D_k D_{k'})^{-1}$, $k \neq k'$ give a new relation

$$n_{2p-1} + n_{2p} = 2n_1. \tag{4.3b}$$

We still have seven parameters n_i, n_{0i}, ρ, γ and five relations leaving two arbitrary parameters. Frequency ρ and wavenumber γ are known from the n_i which satisfy a second relation

$$\gamma = \rho(n_{2p} - n_{2p-1})/2n_1 \qquad \rho = -pn_1 \qquad n_{2p} n_{2p-1} = -(p-1) n_1^2. \tag{4.4}$$

For the construction of the solutions we define $\bar{n}_i = n_i/n_1$, $i = 2p-1$ and $2p$, choose $\bar{n}_{2p-1} = 1 + \sqrt{p}$, $\bar{n}_{2p} = 1 + \sqrt{p}$ and assume for the arbitrary parameters

$$n_{02} > n_{01} > 0. \tag{4.5}$$

We obtain for the other parameters:

$$\sqrt{p}\, n_1 = n_{02} - n_{01} > 0 \qquad n_{2p} = (1/\sqrt{p} - 1)(n_{02} - n_{01}) < 0$$
$$n_{2p-1} = (1 + 1/\sqrt{p})(n_{02} - n_{01}) > 0$$
$$\rho = \sqrt{p}(n_{01} - n_{02}) < 0 \qquad \gamma = -\sqrt{p}\rho > 0 \qquad c = \rho/\gamma = -1/\sqrt{p} \qquad |c| < 1.$$

We still find that the N_i with i odd are sums of positive terms while for i even they are also positive due to their lower bound $n_{02} + n_{2p}/D_j \geq [n_{02} + n_{01}(\sqrt{p} - 1)]/\sqrt{p} > 0$.

We introduce the total mass $M = \Sigma N_i$:

$$M = m_0 + m \sum_1^p 1/D_j$$
$$m_0 = p(n_{01} + n_{02}) > 0 \qquad m = 2pn_1 = 2\sqrt{p}(n_{02} - n_{01}) > 0 \tag{4.6}$$
$$D_j = 1 + d \exp[p(n_{02} - n_{01})(x_j - t/\sqrt{p})] \qquad j = 1, 2, \ldots, p.$$

In the p-dimensional space there exist 2^p sectors corresponding to the different signs of the p coordinates $x_1, \ldots, x_p$. In these sectors the asymptotic limits are p-dimensional manifolds which are the extension of the two-dimensional plateaus of the $4\boldsymbol{v}_i$ model. There exist 2^p such manifolds corresponding to the values m_0, m_0+m (p times), m_0+2m ($p(p-1)/2$ times), $\ldots$, $m_0+(p-1)m$ (p times), m_0+pm (like in a Pascal triangle for the coefficients of $(1+x)^p$). For the time dependence we still find the translation $M(x_1, x_2, \ldots, x_p; t) = M(x_1+ct, x_2+ct, \ldots, x_p+ct; 0)$. Finally all the results of § 3 can be extended. In particular for equidensity manifolds $M =$ constant we can write down x_p as a function of $x_1, \ldots, x_{p-1}$ with an expression which generalises (3.9).

5. Conclusion

In this paper, for the exact multidimensional solutions of the discrete Boltzmann models, two new advances have mainly been obtained. Namely the possibility for 2D solutions to satisfy specular reflection boundary conditions and the construction of 3D solutions. However we have not yet found $(3+1)$-dimensional solutions.

The difficulty for $(3+1)$-dimensional solutions is not yet at the level of positive solutions—it is at the very possibility of building such solutions. Counting arguments, unless miraculous identities occur, are not favourable. However, the present construction of p-dimensional solutions for the hypercubic models is encouraging.

In the previously found $(1+1)$-dimensional solutions satisfying specular reflection (see [4] and references therein), the wall was at $x_1 = 0$ with boundary conditions only for two densities. Here, for the $4\boldsymbol{v}_i$ model, the solutions depend on two spatial coordinates and further the wall at $x_1 = x_2$ requires boundary conditions for all the four densities. The presently constructed 3D solutions, of the Broadwell model with specular reflection, are not the most general ones. However, due to the simplicity of the positivity proof, we have restricted our study to them.

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